

# Measure expanding actions, expanders and warped cones

Federico Vigolo\*

October 20, 2016

## Abstract

We define a way of approximating actions on measure spaces using finite graphs; we then show that in quite general settings these graphs form a family of expanders if and only if the action is expanding in measure. This provides a somewhat unified approach to construct expanders. We also show that the graphs we obtain are uniformly quasi-isometric to the level sets of warped cones. This way we can also prove non-embeddability results for the latter and restate an old conjecture of Gamburd-Jakobson-Sarnak.

## 1 Introduction

In this paper we explore the geometry of finite graphs obtained by approximating measurable actions of groups on metric spaces. As a consequence we construct new families of expanders by approximating measure preserving actions with spectral gaps.

The geometric nature of our construction can be used to shed some light on the geometry underlying some problems related to expanders. In particular, we show that families of expanders are closely related with warped cones and we produce numerous examples of warped cones which coarsely contain a family of expanders and hence do not coarsely embed into any  $L^p$  space.

**Families of expanders.** Expander graphs are graphs at the same time sparse and highly connected. They were first defined in the 70s and they immediately found important applications to applied mathematics and computer science. Soon enough, their interest was recognised also in various fields of pure mathematics, for example in the theory of coarse embeddability of metric spaces. See [HLW06] and [Lub12] for a survey on the subject.

The existence of families of expanders was first proved by Pinsker [Pin73] by probabilistic means. Indeed, he showed that random sequences of graphs with bounded degrees are expanders with high probability. Despite his results, it turned out that defining explicit families of expanders was a challenge. The

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\*The author was funded by the EPSRC Grant 1502483 and the J.T.Hamilton Scholarship. The material is also based upon work supported by the NSF under Grant No. DMS-1440140 while the author was in residence at the MSRI in Berkeley during the Fall 2016 semester.

first explicit examples of expander graphs were built by Margulis [Mar73] using the machinery of Kazhdan's property (T).

To give the precise definition, for a given constant  $\varepsilon > 0$  we say that a graph  $\mathcal{G}$  satisfies an  $\varepsilon$ -linear isoperimetric inequality if every subset of vertices  $W \subset V(\mathcal{G})$  with  $|W| \leq |V(\mathcal{G})|/2$  satisfies  $|\partial W| \geq \varepsilon|W|$ . Here  $\partial W$  denotes the exterior vertex boundary *i.e.* the set of vertices in  $V(\mathcal{G}) \setminus W$  linked by an edge to vertices in  $W$ . The largest constant  $\varepsilon$  so that  $\mathcal{G}$  satisfies an  $\varepsilon$ -linear isoperimetric inequality is known as the *Cheeger constant* of  $\mathcal{G}$ . Note that in literature different kinds of boundaries are often used, but this is not an issue because they are all coarsely equivalent when the considered graphs have bounded degree.

A *family of expanders* is a sequence of finite graphs  $\mathcal{G}_n$  of uniformly bounded degree with  $|V(\mathcal{G}_n)| \rightarrow \infty$  such that they all satisfy an  $\varepsilon$ -linear isoperimetric inequality where the constant  $\varepsilon > 0$  is fixed. Notice that every finite graph satisfies an isoperimetric inequality with positive  $\varepsilon$  as long as it is connected, thus the important bit here is the existence of a constant  $\varepsilon > 0$  independent of  $n$ . In this sense, the above is a notion of strong connectedness.

Various different constructions of expanders are known nowadays. They are generally built either using zig-zag products, or techniques from additive combinatorics, or infinite dimensional representation theory. The main goal of this work is to illustrate a general procedure to build families of expanders out of actions on measure spaces, partially expanding and linking the latter two techniques. Our approach is rather geometric in nature and it turns out to be fruitful in the study of warped cones as well.

**Coarse embeddings and warped cones.** Since the late 50's, many mathematicians have been interested in studying different notions of embeddings of metric spaces into Hilbert spaces. It was known that every separable metric space is homeomorphic to a subset of  $L^2(0, 1)$ , but in [Enf70] Enflo provided an example of a separable metric space which does not embed into a Hilbert space *uniformly*. It was later asked by Gromov [Gro93, p218] whether every separable metric space embeds into a Hilbert space *coarsely*.

It was proved by Yu in [Yu00] that the coarse Baum-Connes (and hence the Novikov conjecture) holds true for every metric spaces which coarsely embed into a Hilbert space. Yu's result combined with Gromov's question provided a valid strategy to tackle such conjectures, but in [DGLY02] it was built an example of a separable metric space that does not coarsely embed into a Hilbert space. Their example was constituted by a family of locally finite graphs with growing degrees and therefore it did not have *bounded geometry* (recall that a metric space  $X$  has bounded geometry if for every  $\varepsilon, R > 0$  there exists an  $N \in \mathbb{N}$  such that any  $\varepsilon$ -separated subset of an  $R$ -ball of  $X$  has at most  $N$  elements).

It was pointed out by Gromov that the reason why those graphs could not coarsely embed was related with their expansion properties and he also noted that families of expanders do not coarsely embed into Hilbert spaces. This idea led Higson to produce a counterexample to the coarse Baum-Connes conjecture [Hig99]. In [Gro03], Gromov used probabilistic methods to build finitely generated groups that contain a coarsely embedded family of expanders and therefore do not coarsely embed into Hilbert spaces. These groups were used in [HLS02] to produce counterexamples to the Baum-Connes conjecture. It is in this context that John Roe first defined the warped cones.

Following [Roe05], let  $(M, \varrho)$  be a compact Riemannian manifold and let a finitely generated group  $\Gamma = \langle S \rangle$  act by diffeomorphisms on  $M$ . The *warped cone* associated with this action is the metric space  $(\mathcal{O}_\Gamma(M), \delta_\Gamma)$  obtained from the infinite Riemannian cone  $(M \times [1, \infty), t^2 \varrho + dt^2)$  by warping the metric, imposing the condition that the distance between any two points of the form  $(x, t)$  and  $(s \cdot x, t)$  is at most 1. Warped cones are spaces with bounded geometry and with a very large group of translations, but their most interesting features regard their coarse geometry (which does not depend on the choice of the generating set). Indeed, Roe showed that this construction is flexible enough to produce examples of spaces with or without property A of Yu: warped cones of amenable actions have property A and, for a particular kind of actions, the converse is also true. Moreover, warped cones also provide further examples of spaces that do not coarsely embed into Hilbert spaces.

The counter examples to the coarse Baum-Connes conjecture built in [Hig99, HLS02] are based on the existence, for a coarse disjoint union of expanders  $X$ , of non-compact ghost projections for the Roe algebra  $C^*(X)$ . In [DN15] it is shown that also warped cones can have such ghost projections. This means that they are good candidates for a new class of counterexamples to the coarse Baum-Connes conjecture. To the present day, the only known counterexamples to Baum-Connes are constructed using expanders, it is therefore of interest to understand how families of expanders relate to warped cones.

If  $\Gamma$  is a finitely generated subgroup of a compact Lie group  $G$  one can construct a warped cone  $\mathcal{O}_\Gamma(G)$  by considering the action by left multiplication. In [Roe05] it is proved that, in this setting, if the warped cone  $\mathcal{O}_\Gamma(G)$  coarsely embeds in a Hilbert space then the group  $\Gamma$  must satisfy an analytical weak version of amenability: the Haagerup property. It is a natural question to ask the following:

**Question 1.1.** *Is it true that if  $\Gamma$  has the Haagerup property then any warped cone  $\mathcal{O}_\Gamma(M)$  coarsely embeds into a Hilbert space?*

As a consequence of our result we can give a negative answer to this question. Indeed, there exist many warped cones for  $G$  over a non-abelian free group  $\Gamma$  which cannot be embedded into a Hilbert space for the strongest of reasons: they contain a family of expanders.

**Expanders via approximating graphs and other results.** Given a finitely presented group  $\Gamma$  with a finite generating set  $S$  and a measurable action on a probability space  $\Gamma \curvearrowright (X, \nu)$ , one can try to ‘approximate’ this action *via* a finite graph. Specifically, if we choose a finite partition  $\mathcal{P}$  of  $X$  into measurable subsets, then we can define an *approximating graph* by considering the graph  $\mathcal{G}(\mathcal{P})$  whose vertices are the regions of the partition and such that two regions  $R, R' \in \mathcal{P}$  are linked by a vertex if there exists an element  $s$  in the generating set  $S$  so that the image  $s(R)$  intersect  $R'$  non-trivially.

We say that such an action is *expanding in measure* if there exists a constant  $\alpha > 0$  so that for every measurable set  $A \subset X$  with  $\nu(A) < 1/2$  the union of the images  $s(A)$  with  $s \in S$  has measure at least  $(1 + \alpha)\nu(A)$ . Our main tool will then be the following:

**Theorem A.** *Let  $(X, d)$  be a locally compact metric space and  $\nu$  a Radon measure thereon. Moreover, let  $\Gamma = \langle S \rangle$  be a finitely generated group with a continuous action  $\Gamma \curvearrowright (X, d)$  which does not ‘distort the measure  $\nu$  too much’.*

Assume that  $\mathcal{P}_n$  is a family of measurable partitions of  $X$  which are ‘regular enough’ and so that the diameters of most of the regions tend to zero. Then the approximating graphs  $\mathcal{G}(\mathcal{P}_n)$  all satisfy an isoperimetric inequality with the same Cheeger constant  $\varepsilon > 0$  if and only if the action is expanding in measure.

We refer the reader to Section 3 and Theorem 3.10 for precise statements and definitions.

During the proof of Theorem A it is shown that the Cheeger constant  $\varepsilon$  and the expansion constant  $\alpha$  depend rather explicitly on each other. Moreover, the same statement holds true also when the graphs and the total measure  $\nu(X)$  are infinite.

*Remark 1.2.* It was already known that when a graph comes naturally from an action on measure spaces then one can estimate its Cheeger constant by studying expanding properties of the action. In fact, this has been used more or less implicitly in many works on expanders (e.g. [Mar73, GG81, Sha97]). Despite this, to our knowledge nobody actually made explicit an equivalence such as that of Theorem A.

It appears to us, that our method is well suited to constructing numerous families of expanders. We also believe that the intrinsic geometric nature of the approximating graphs can be successfully used to provide some intuition in settings whose nature could otherwise be rather obscure. An excellent example in this sense is given by the work of Bourgain and Yehudayoff: in [BY13] they managed to build the first known families of *monotone expanders* using techniques quite similar to ours.

As a further example of the convenience of our geometric intuition, we can use the geometric control we have on our construction to find a close connection between approximating graphs and warped cones. Indeed, let  $\Gamma$  act by diffeomorphisms on a compact Riemannian manifold  $M$  and let  $\mathcal{O}_\Gamma(M)$  be the resulting warped cone. The coarse structure of the level sets of the warped cone  $M \times \{t\} \subset \mathcal{O}_\Gamma(M)$  with respect to the restriction of the warped metric  $\delta_\Gamma$  is actually equal to the coarse structure of some graphs approximating the action  $\Gamma \curvearrowright M$ . As a consequence, we prove the following:

**Theorem B.** *Let  $(\mathcal{O}_\Gamma(M), \delta_\Gamma)$  be the warped cone built from an action of a finitely generated group  $\Gamma$  on a compact Riemannian manifold  $M$ . Then for every sequence  $t_n \rightarrow \infty$  the level sets  $(M \times \{t_n\}, \delta_\Gamma)$  are uniformly quasi-isometric to a sequence of expander graphs if and only if the action  $\Gamma \curvearrowright M$  is expanding in measure.*

Since it is known that expander graphs do not coarsely embed into any  $L^p$  space (see [Mat97]), we get as a corollary that if the  $\Gamma$ -action is expanding in measure the warped cone  $\mathcal{O}_\Gamma(M)$  does not embed into any  $L^p$  space either. This also provides an alternative proof of the main result of [NS15].

Theorem B gives a strongly negative answer to Question 1.1. Indeed, there are numerous examples of actions of non-abelian free groups which are expanding in measure (see [BG07] and [dC08]), thus the warped cone does not embed in any Hilbert space. Nevertheless, free groups do have the Haagerup property.

Concrete examples may be produced using the fact that a measure preserving action on a probability space  $\Gamma \curvearrowright (X, \nu)$  is expanding in measure if and only

if the induced unitary representation  $\pi: \Gamma \curvearrowright L^2(X)$  has a spectral gap. In particular, given a measure preserving action of a locally compact group on probability space  $G \curvearrowright (X, \nu)$ , it follows that its restriction to a finitely generated subgroup  $\Gamma = \langle S \rangle < G$  is expanding in measure if and only if  $S$  is a Kazhdan set for the representation  $G \curvearrowright L^2(X)$  (see Definition 8.1).

The spectral properties of unitary representations are a quite well understood and many actions are known to produce spectral gaps (see *e.g.* [BG07, BdS14, CG11, Sha00, Bek03, GJS99]). For example, if  $a, b \in \mathrm{SO}(3, \mathbb{Q})$  are two matrices with algebraic coefficients and generate a non-abelian free group, then their action by rotations on  $\mathbb{S}^2$  has spectral gap [BG07]. It follows that the level sets of the warped cone  $\mathcal{O}_{(a,b)}(\mathbb{S}^2)$  are uniformly quasi-isometric to a family of expander graphs.

In a more general setting, let  $G$  be a compact simple Lie group,  $(g_1, \dots, g_k) \in G^k$  random  $k$ -tuple and  $\Gamma(g_1, \dots, g_k) < G$  the generated subgroup. The following is an open conjecture:

**Conjecture 1.3.** *Let  $k \geq 2$ . For almost every  $k$ -tuple the action  $\Gamma(g_1, \dots, g_k) \curvearrowright G$  by left multiplication has a spectral gap.*

Using Theorem B and the spectral criterion for expansion, we can restate this conjecture in the language of warped cones:

**Theorem C.** *Conjecture 1.3 holds true if and only if for almost every  $k$ -tuple with  $k \geq 2$  one (any) unbounded sequence of level sets of the warped cone  $\mathcal{O}_{\Gamma(g_1, \dots, g_n)}(G)$  forms a family of expanders.*

**Organisation of the paper.** In Section 2 we introduce some basic facts and notation that we use throughout the paper and in Section 3 we prove Theorem A. In Section 4 we describe reasonably flexible conditions under which it is possible to prove that approximating graphs of actions on metric spaces have uniformly bounded degrees. In Section 5 we introduce the Voronoi tessellations as a fairly general way of constructing regular partitions. These partitions will then be used in Section 6, where we define warped cones and we investigate their relations with the approximating graphs, hence proving Theorem B.

In Section 7 we prove the spectral criterion for measure preserving actions that characterises the expanding actions as those actions that have a spectral gap. This criterion is then used extensively in Section 8, where we explain how to adapt the formalism of Kazhdan sets to the setting of expanding actions. We then review some well-known results of representation theory and we provide numerous examples of expanding actions and other applications.

**Acknowledgements.** I wish to thank my supervisor Cornelia Druţu for her advice and encouragement and for carefully reading this work. I also wish to thank Yves Benoist for useful conversations and Gareth Wilkes for his helpful suggestions and comments.

## 2 Some notation and preliminary results

Throughout the paper,  $\mathcal{G}$  will denote a simplicial non-oriented graph, possibly with unbounded degree (a graph has *degree*  $D$  if every vertex is contained in at

most  $D$  edges). We wish to remark that we only use graphs with unbounded degree for the sake of generality, so that the statements of the results in Section 3 do not require further hypotheses. All the other sections will only involve graphs with bounded degree.

For every graph  $\mathcal{G}$  and every set of vertices  $W \subset V(\mathcal{G})$  we will denote by  $\partial W$  the *external vertex boundary*

$$\partial W := \{v \in V(\mathcal{G}) \setminus W \mid \exists w \in W \text{ s.t. } (v, w) \text{ is an edge}\}.$$

We define the Cheeger constant of the graph  $\mathcal{G}$  as the infimum

$$h(\mathcal{G}) := \inf \left\{ \frac{|\partial W|}{|W|} \mid W \subset V(\mathcal{G}) \text{ finite, } |W| \leq \frac{1}{2}|V(\mathcal{G})| \right\}$$

(if the graph is infinite the condition on the cardinality of the set  $W$  is vacuous).

**Definition 2.1.** A sequence of finite graphs  $\mathcal{G}_n$  is a *family of expanders* if  $|V(\mathcal{G}_n)| \rightarrow \infty$  and there are two constants  $C, \varepsilon > 0$  so that every graph  $\mathcal{G}_n$  has degree bounded above by  $C$  and Cheeger constant  $h(\mathcal{G}_n)$  at least  $\varepsilon$ .

*Remark 2.2.* In the literature the Cheeger constant is usually defined using the edge-boundary, we chose to use this less standard definition because it is notationally more convenient for our purposes. The two notions are coarsely equivalent when dealing with graphs with bounded degree.

Most of our results will be better understood in the context of coarse geometry. Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.

**Definition 2.3.** A map  $f: (X, d_X) \rightarrow (Y, d_Y)$  is a *quasi-isometry* if there are two positive constants  $L, A \geq 0$  so that for every couple  $x, x' \in X$  we have

$$\frac{1}{L}d_X(x, x') - A \leq d_Y(f(x), f(x')) \leq Ld_X(x, x') + A$$

and for every  $y \in Y$  there exists a  $x \in X$  with  $d_Y(y, f(x)) \leq A$ . Such a function is an  $(L, A)$ -*quasi-isometry*.

Two spaces are *quasi-isometric* if there exists a quasi-isometry between them. It is easy to show that being quasi-isometric is an equivalence relation. More generally, two families of metric spaces  $(X_n, d_{X_n}), (Y_n, d_{Y_n})$  are *uniformly quasi-isometric* if there exists a family of quasi-isometries  $f_n: (X_n, d_{X_n}) \rightarrow (Y_n, d_{Y_n})$  which share the same constants  $L$  and  $A$ .

Connected graphs can be seen as metric spaces when endowed with their path metric. The following Lemma is well known and it can be proved with easy but tedious counting arguments:

**Lemma 2.4.** *If two families of graphs  $\mathcal{X}_n$  and  $\mathcal{Y}_n$  with uniformly bounded degree are uniformly quasi-isometric, then one of them is a family of expanders if and only if the other is.*

With an abuse of notation, we give the following:

**Definition 2.5.** A sequence of metric spaces  $(X_n, d_n)$  *forms a family of expanders* if it is uniformly quasi-isometric to a family of expander graphs  $\mathcal{X}_n$ .

*Remark 2.6.* Note that Lemma 2.4 implies that a family of graphs with uniformly bounded degrees form a family of expanders in the sense of Definition 2.5 if and only if it is a genuine family of expander graphs (Definition 2.1).

One of the many remarkable features of expander graphs is that it is not possible to embed them into any Hilbert space without greatly distorting the metric. More precisely, recall the following:

**Definition 2.7.** A map between metric spaces  $f: (X, d_X) \rightarrow (Y, d_Y)$  is a *coarse embedding* if there are two unbounded functions  $\eta_-, \eta_+: [0, \infty) \rightarrow [0, \infty)$  so that

$$\eta_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \eta_+(d_X(x, x'))$$

for every  $x, x' \in X$ . We say that a family of metric spaces  $(X_n, d_{X_n})$  *coarsely embeds* in  $(Y, d_Y)$  if there are coarse embeddings  $f_n: (X_n, d_{X_n}) \rightarrow (Y, d_Y)$  all with the same bounding functions  $\eta_-$  and  $\eta_+$  (we are dropping the word ‘uniform’ from the terminology).

It is a well known fact that a family of expanders does not coarsely embed into any  $L^p$  space for  $1 \leq p < \infty$  [Mat97] (see also [Oza04, Appendix A]).

Throughout the paper we will denote by  $\Gamma$  a finitely generated group and  $S$  will always denote a finite symmetric generating set containing the identity element:  $1 \in S$ ,  $S = S^{-1}$ .

We will generally consider measurable actions of  $\Gamma$  on a measure space  $(X, \nu)$  (*i.e.* actions such that  $g: X \rightarrow X$  is measurable for every  $g \in \Gamma$ ). For every subset  $A \subseteq X$  we will always use the notation  $S \cdot A$  to denote the union of the images of  $A$  under the elements of  $S$

$$S \cdot A := \bigcup_{s \in S} s \cdot A.$$

The requirement  $1 \in S$  is mainly made out of convenience, so that we always have  $A \subseteq S \cdot A$ .

### 3 Measure expansion and Cheeger constants

The key notion in our study is the following:

**Definition 3.1.** A measurable action  $\rho: \Gamma \curvearrowright (X, \nu)$  is *expanding in measure* if there exists a constant  $\alpha > 0$  such that  $\nu(S \cdot A) \geq (1 + \alpha)\nu(A)$  for every measurable set  $A \subset X$  with finite measure and  $\nu(A) \leq \nu(X)/2$  (the latter condition is vacuous when  $X$  has infinite measure). When this is the case, we say that the action is  $\alpha$ -*expanding*.

*Remark 3.2.* Our notion of expansion is not quite new in the literature. In the notation of [GMP16], the action  $\rho$  is expanding in measure if  $X$  is a *domain of expansion* for it. In [BY13] such a  $\rho$  is said to be a *continuous expander* (if the action is also differentiable).

We define a *measurable partition* of  $(X, \nu)$  to be a countable family of disjoint subsets (regions)  $\mathcal{P} = \{R_i \mid i \in I\}$  such that

$$\nu\left(X \setminus \prod_{i \in I} R_i\right) = 0.$$

We can now define the key object of our study:

**Definition 3.3.** Given a measurable action  $\rho: \Gamma \curvearrowright X$  and a measurable partition  $\mathcal{P} = \{R_i \mid i \in I\}$  of  $(X, \nu)$ , their *approximating graph* is the (non oriented) graph  $\mathcal{G}_\rho(\mathcal{P})$  whose set of vertices is the set of regions  $V(\mathcal{G}_\rho(\mathcal{P})) := \mathcal{P}$  and such that the couple  $(R_i, R_j)$  is an edge if and only if there exists an element  $s \in S$  with

$$\nu((s \cdot R_i) \cap R_j) \neq 0.$$

When there is no risk of confusing the action  $\rho$ , we will drop it from the notation and simply denote the approximating graph by  $\mathcal{G}(\mathcal{P})$ .

For the graph  $\mathcal{G}_\rho(\mathcal{P})$  to give any interesting information on the dynamical system, we need to require some tameness conditions on the action itself and on the partition considered. The most important among such requirements is some kind of control on the ratios of the measures of the regions of the partition  $\mathcal{P}$ .

**Definition 3.4.** A partition  $\mathcal{P}$  has *bounded measure ratios* if the measure of every region  $R \in \mathcal{P}$  is finite and there exists a constant  $Q \geq 1$  such that for every couple of regions  $R_i, R_j$  in  $\mathcal{P}$  one has

$$\frac{1}{Q} \leq \frac{\nu(R_i)}{\nu(R_j)} \leq Q.$$

The fundamental observation is the following simple lemma:

**Lemma 3.5.** *Let  $\rho: \Gamma \curvearrowright (X, \nu)$  be an  $\alpha$ -expanding action. For every partition  $\mathcal{P}$  with measure ratios bounded by a constant  $Q$ , the approximating graph  $\mathcal{G}_\rho(\mathcal{P})$  has Cheeger constant bounded away from zero*

$$h(\mathcal{G}_\rho(\mathcal{P})) \geq \varepsilon > 0$$

and the constant  $\varepsilon = \varepsilon(\alpha, Q)$  depends only on the expansion parameter  $\alpha$  and the bound on measure ratios  $Q$ .

*Proof.* Let  $W$  be any finite set of vertices of  $\mathcal{G}_\rho(\mathcal{P})$  with  $|W| \leq |\mathcal{G}_\rho(\mathcal{P})|/2$  and consider the measurable set

$$A := \coprod_{R_i \in W} R_i.$$

Up to measure zero sets we have

$$S \cdot A \subseteq \bigcup \left\{ R_i \mid \nu((S \cdot A) \cap R_i) > 0 \right\} = \coprod_{W \cup \partial W} R_i = B.$$

If  $\nu(A) \leq \nu(X)/2$  then

$$\nu(B) \geq (1 + \alpha)\nu(A).$$

and since  $\mathcal{P}$  has bounded ratios we conclude

$$Q|\partial W| \left( \inf_{R \in \mathcal{P}} \nu(R) \right) \geq \nu(B \setminus A) \geq \alpha\nu(A) \geq \alpha|W| \left( \inf_{R \in \mathcal{P}} \nu(R) \right)$$

whence

$$\frac{|\partial W|}{|W|} \geq \frac{\alpha}{Q}.$$



On the other hand, if  $\nu(A) > \nu(X)/2$  let  $C := X \setminus (S \cdot A)$  and notice that  $(S \cdot C) \setminus C \subseteq (S \cdot A) \setminus A$  because  $S = S^{-1}$ . Then we have:

$$\nu((S \cdot A) \setminus A) \geq \alpha \nu(C) = \alpha (\nu(X) - \nu(A) - \nu((S \cdot A) \setminus A))$$

whence

$$\nu(B \setminus A) \geq \nu((S \cdot A) \setminus A) \geq \frac{\alpha}{1 + \alpha} (\nu(X) - \nu(A)).$$

Since  $|W| \leq |\mathcal{P}|/2$ , by the bound on measure ratios we get

$$\nu(X \setminus A) \geq \frac{1}{Q} \nu(A).$$

Using the same argument as above and combining the inequalities so obtained we conclude that

$$\frac{|\partial W|}{|W|} \geq \min \left\{ \frac{\alpha}{Q}, \frac{\alpha}{(1 + \alpha)Q^2} \right\}$$

as desired.  $\square$

**Corollary 3.6.** *Let  $(X, \nu)$  be a probability space and  $\Gamma \curvearrowright (X, \nu)$  an action expanding in measure. Assume we are given a sequence of finite measurable partitions  $\mathcal{P}_n$  with  $|\mathcal{P}_n| \rightarrow \infty$  and measure ratios uniformly bounded by the same constant  $Q$ . Then the sequence of approximating graphs  $\mathcal{G}(\mathcal{P}_n)$  forms a family of expanders if and only if they have uniformly bounded degree.*

In certain situations, considering finer and finer measurable partitions on the same dynamical system  $\Gamma \curvearrowright (X, \nu)$  can yield a converse to Lemma 3.5. Recall that a measure  $\nu$  on the Borel  $\sigma$ -algebra of a Hausdorff topological space is a *Radon measure* if it is locally finite (every point has a neighbourhood of finite measure) and inner regular (the measure of every measurable set  $A \subseteq X$  is equal to the supremum of the measures of its compact subsets  $K \subseteq A$ ).

**Definition 3.7.** Given a sequence  $\mathcal{P}_n$  of measurable partitions of  $X$  we say that the radius of the tiles *tends to zero in measure on compact sets* if there exists a sequence of numbers  $r_n > 0$  with  $r_n \rightarrow 0$  such that for

$$E_n := \bigcup \{R \mid R \in \mathcal{P}_n, \text{diam}(R) > r_n\}$$

we have  $\nu(E_n \cap K) \rightarrow 0$  for every compact set  $K \subseteq X$ .

**Proposition 3.8.** *Let  $(X, d, \nu)$  be a locally compact metric space with a Radon measure thereof. Let  $\mathcal{P}_n$  be a sequence of measurable partitions of  $X$  with uniformly bounded measure ratios and such that the radius of the tiles tends to zero in measure on compact sets. Then, for any continuous action  $\Gamma \curvearrowright X$  the existence of a uniform positive Cheeger constant  $\varepsilon > 0$  for the approximating graphs  $\mathcal{G}(\mathcal{P}_n)$  implies that the action is expanding in measure.*

*Proof.* Since  $\nu$  is a Radon measure, it is enough to prove that there is a positive constant  $\alpha > 0$  so that for every compact set  $K \subset X$  with  $\nu(K) \leq \nu(X)/2$  we have  $\nu(S \cdot K) \geq (1 + \alpha)\nu(K)$ .

For any set  $A \subseteq X$  we will denote by  $V_n(A) \subset X$  the set of cells in  $\mathcal{P}_n$  whose intersection with  $A$  has positive measure

$$V_n(A) := \{R \in \mathcal{P}_n \mid \nu(R \cap A) > 0\}.$$

and denote by  $\mathcal{N}_n(A)$  their union

$$\mathcal{N}_n(A) := \bigcup_{R \in \mathcal{V}_n(A)} R \subseteq X.$$

Notice that  $\mathcal{N}_n(K \setminus E_n) = \mathcal{N}_n(K) \setminus E_n \subseteq N_{r_n}(K) \setminus E_n$  where  $N_{r_n}$  denotes the neighbourhood of radius  $r_n$  and  $\{r_n\}_{n \in \mathbb{N}}$  is a sequence as in Definition 3.7.

Since the measure of the unbounded tiles  $E_n$  tends to zero on every compact set we have:

$$\nu(K) = \lim_{n \rightarrow \infty} \nu(K \setminus E_n) \leq \lim_{n \rightarrow \infty} \nu(\mathcal{N}_n(K \setminus E_n)) \leq \lim_{n \rightarrow \infty} \nu(N_{r_n}(K))$$

and since the last term is equal to  $\nu(K)$  all of the above are actually equalities.

Since  $X$  is locally compact, there exists a radius  $r$  small enough so that  $N_r(K)$  is still compact. As the action of  $\Gamma$  is continuous, we have that the restrictions  $s: N_r(K) \rightarrow X$  with  $s \in S$  are uniformly continuous. Since  $r_n \rightarrow 0$  by hypothesis, it follows that there exists a sequence of numbers  $r'_n$  such that  $s \cdot N_{r_n}(K) \subseteq N_{r'_n}(s \cdot K)$  for every  $s \in S$  and  $r'_n \rightarrow 0$ .

Observing that

$$\mathcal{N}_n(S \cdot \mathcal{N}_n(K \setminus E_n)) \subseteq N_{r_n + r'_n}(S \cdot (K \setminus E_n))$$

and using an argument similar to the above we can conclude that

$$\nu(S \cdot K) = \lim_{n \rightarrow \infty} \nu(\mathcal{N}_n(S \cdot \mathcal{N}_n(K \setminus E_n))),$$

whence

$$\frac{\nu(S \cdot K)}{\nu(K)} = \lim_{n \rightarrow \infty} \frac{\nu(\mathcal{N}_n(S \cdot \mathcal{N}_n(K \setminus E_n)))}{\nu(\mathcal{N}_n(K \setminus E_n))} \quad (1)$$

By hypothesis the partitions  $\mathcal{P}_n$  have uniformly bounded measure ratios. That is, there exist a constant  $Q$  such that for any  $n$  and any  $R, R' \in \mathcal{P}_n$  we have  $\nu(R) \leq Q\nu(R')$ . Noticing that for any set  $A \subseteq X$  we have  $\mathcal{V}_n(S \cdot A) = \mathcal{V}_n(A) \amalg \partial \mathcal{V}_n(A)$ , we get

$$\frac{\nu(\mathcal{N}_n(S \cdot A))}{\nu(\mathcal{N}_n(A))} = 1 + \frac{\nu(\mathcal{N}_n(S \cdot A) \setminus \mathcal{N}_n(A))}{\nu(\mathcal{N}_n(A))} \geq 1 + \frac{|\partial \mathcal{V}_n(A)|}{|\mathcal{V}_n(A)|} Q^{-1}$$

and if we apply this inequality to  $A = K \setminus E_n$ , equation (1) becomes

$$\frac{\nu(S \cdot K)}{\nu(K)} \geq 1 + \lim_{n \rightarrow \infty} \frac{|\partial \mathcal{V}_n(K \setminus E_n)|}{|\mathcal{V}_n(K \setminus E_n)|} Q^{-1}.$$

At this point it is enough to find a uniform bound  $\alpha > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{|\partial \mathcal{V}_n(A)|}{|\mathcal{V}_n(A)|} \geq \alpha$$

for any set with measure  $\nu(A) \leq \nu(X)/2$ . By approximating if necessary, without loss of generality assume  $\nu(A) < \nu(X)/2$ , so that for  $n$  large enough we also have  $\nu(\mathcal{N}_n(A)) < \nu(X)/2$ .

If  $|V_n(A)|$  is lesser than or equal to  $|V_n(X)|/2$  then

$$\frac{|\partial V_n(A)|}{|V_n(A)|} \geq \varepsilon$$

by definition of Cheeger constant. If this is not the case, we need to use an argument similar to that of Lemma 3.5. That is, denote by  $W_n$  the complement set  $W_n := V_n(X) \setminus V_n(A)$  and notice that

$$\partial V_n(A) = \partial_{\text{int}}(W_n) \supseteq \partial(W_n \setminus \partial_{\text{int}}(W_n)),$$

where  $\partial_{\text{int}}(W_n)$  denotes the interior vertex boundary, *i.e.* the set of vertices of  $W_n$  which are endpoints of edges with one endpoint in the complement  $V_n(X) \setminus W_n$ . It follows that

$$|\partial V_n(A)| \geq \varepsilon |W_n \setminus \partial_{\text{int}}(W_n)| = \varepsilon (|W_n| - |\partial V_n(A)|).$$

Again, by the bound on measure ratios we have

$$|V_n(A)| \leq Q|W_n|,$$

thus

$$|\partial V_n(A)| \geq \frac{\varepsilon}{1 + \varepsilon} |W_n| \geq \frac{\varepsilon}{Q(1 + \varepsilon)} |V_n(A)|$$

as desired.  $\square$

*Remark 3.9.* The proof of Proposition 3.8 can be somewhat simplified if one assumes that the diameters of all the tiles of  $\mathcal{P}_n$  tend to zero. We decided to provide a more general statement so that it could be applied to metric spaces with cusps as well.

We can combine Proposition 3.8 with Lemma 3.5 to prove Theorem A. Specifically, we have the following:

**Theorem 3.10.** *Let  $\Gamma \curvearrowright (X, d, \nu)$  be a continuous action and  $\mathcal{P}_n$  a family of finite measurable partitions of  $X$  with uniformly bounded measure ratios and such that the radius of the tiles tends to zero in measure on compact sets. Then the action is expanding in measure if and only if all the approximating graphs share a common lower bound on their Cheeger constant.*

*Remark 3.11.* Notice that if the approximating graphs are already known to share a uniform bound on their degrees, then one can modify the proofs of Lemma 3.5 and Proposition 3.8 in order to extend them to measurable actions of semigroups.

## 4 Metric bounds on degrees

Using the tools of Section 3 we can now build families of graphs with uniform lower bounds on their Cheeger constants. Still, to build examples of expanders we also need to find a way to bound the degrees of the approximating graphs. One way of doing it is by using additional metric structures.

Let  $(X, d, \nu)$  be a metric space with a Borel measure  $\nu$ . Such measure is *doubling* if there exists a constant  $D$  such that

$$\nu(B_x(2r)) \leq D\nu(B_x(r))$$

for every  $x \in X$  and every radius  $r > 0$ .

For any bounded subset  $A \subset (X, d)$  we will define its *eccentricity* as

$$\xi(A) := \sup \left\{ \frac{r}{R} \mid \exists x \in X, B_x(r) \subseteq (A) \subseteq B_x(R) \right\}.$$

Recall that, given an increasing homeomorphism  $\eta: [0, +\infty) \rightarrow [0, +\infty)$ , a homeomorphism  $f: X \rightarrow X$  is  $\eta$ -*quasi-symmetric* if

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta \left( \frac{d(x, y)}{d(x, z)} \right)$$

for every choice of points  $z \neq x \neq y$  in  $X$ . We say that  $f$  is *quasi-symmetric* if it is  $\eta$ -quasi-symmetric for some  $\eta: [0, \infty) \rightarrow [0, \infty)$ .

We will also say that a measurable map  $f: (X, \nu) \rightarrow (X, \nu)$  has *measure distortion* bounded by  $\Theta \geq 1$  if

$$\frac{1}{\Theta} \nu(A) \leq \nu(f(A)) \leq \Theta \nu(A)$$

for every measurable set  $A \subseteq X$ .

An action  $\rho: G \curvearrowright (X, d)$  is quasi-symmetric (respectively, has bounded measure distortion) if  $\rho(g)$  is a quasi-symmetric homeomorphism (respectively, has bounded measure distortion) for every  $g \in G$ . We do not require the bounds to be uniform.

**Proposition 4.1.** *Let  $(X, d, \nu)$  be a doubling measure space and  $\mathcal{P}$  be a measurable partition with measure ratios bounded by  $Q \geq 1$  and such that all the regions  $R \in \mathcal{P}$  are bounded and have uniformly bounded eccentricity  $\xi(R) \leq \xi$  for some constant  $\xi > 0$ .*

*Given a quasi-symmetric action  $\rho: \Gamma \curvearrowright (X, d, \nu)$  with bounded measure distortion, let  $\eta: [0, \infty) \rightarrow [0, \infty)$  and  $\Theta \geq 1$  be such that for every generator  $s \in S$ , the homeomorphism  $\rho(s)$  is  $\eta$ -quasi-symmetric and has measure distortion bounded by  $\Theta$ .*

*Then the approximating graph  $\mathcal{G}(\mathcal{P})$  has bounded degree and this bound depends only on  $\eta, \Theta, Q, \xi$  and the doubling constant  $D$ .*

*Proof.* Fix any  $s \in S$  and, for any region  $R \in \mathcal{P}$ , let

$$J := \{R_i \in \mathcal{P} \mid R_i \cap s(R) \neq \emptyset\}.$$

It is enough to prove that there is a uniform bound on  $|J|$ .

Since  $s$  is  $\eta$ -quasi-symmetric, the image  $s(R)$  has eccentricity at most  $\eta(\xi)$ . Thus there are  $x \in X$  and  $0 < r_1 \leq r_2$  with  $r_2 \leq \eta(\xi)r_1$  such that

$$B_x(r_1) \subseteq s(R) \subseteq B_x(r_2).$$

We can then bound the measure  $\nu(B_x(r_2))$  due to the doubling condition on  $X$

$$\nu(B_x(r_2)) \leq D^{\lceil \log_2(r_2/r_1) \rceil} \nu(B_x(r_1)) \leq D^{\lceil \log_2(\eta(\xi)) \rceil} \nu(s(R))$$

where  $\lceil t \rceil$  denotes the smallest integer  $k \geq t$ . Letting  $L_1 = D^{\lceil \log_2(\eta(\xi)) \rceil}$  we get

$$\nu(B_x(r_2)) \leq L_1 \Theta \nu(R)$$

by bounded measure distortion.

Let  $r_3$  be the smallest real number such that  $R_i \subseteq B_x(r_2 + r_3)$  for every region  $R_i \in J$ . By definition, there exists a region  $R_j$  intersecting  $s(R)$  non trivially and having diameter  $\text{diam}(R_j) \geq r_3$ . Thus there exists  $y \in X$  with  $B_y(r_3/\xi) \subseteq R_j$ .

Notice that

$$\nu\left(\prod_{i \in J} R_i\right) \leq \nu(B_y(2r_3 + r_2)) \leq D^{\lceil \log_2\left(\frac{(2r_3 + r_2)\xi}{r_3}\right) \rceil} \nu(B_y(r_3/\xi)).$$

Letting  $L_2(t) := D^{\lceil \log_2((2+t)\xi) \rceil}$  yields

$$\frac{|J|}{Q} \nu(R_j) \leq \nu(B_y(2r_3 + r_2)) \leq L_2(r_2/r_3) \nu(B_y(r_3/\xi)) \leq L_2(r_2/r_3) \nu(R_j).$$

On the other hand we have

$$\nu\left(\prod_{i \in J} R_i\right) \leq \nu(B_x(r_3 + r_2)) \leq D^{\lceil \log_2\left(\frac{(r_3 + r_2)}{r_2}\right) \rceil} \nu(B_x(r_2)),$$

thus letting  $L_3(t) = D^{\lceil \log_2(t+1) \rceil}$  we get

$$\frac{|J|}{Q} \nu(R) \leq L_3(r_3/r_2) \nu(B_x(r_2)) \leq L_3(r_3/r_2) L_1 \Theta \nu(R).$$

Thus we conclude

$$\begin{aligned} |J| &\leq \min \left\{ QL_2(r_2/r_3), Q\Theta L_3(r_3/r_2) L_1 \right\} \\ &\leq \sup_{t>0} \left( \min \left\{ QL_2(t), Q\Theta L_3\left(\frac{1}{t}\right) L_1 \right\} \right) \end{aligned}$$

and the latter is bounded.  $\square$

Proposition 4.1 is fairly general in that it deals with a large class of maps and measure spaces. Still, at times its hypotheses might be cumbersome to work with. It is especially so when one already knows that the action and the tessellation already satisfy much stronger hypotheses that make some of the requirements of Proposition 4.1 redundant. For example, it is easy to prove the following:

*Remark 4.2.* Let  $(X, d)$  be a metric space with a partition  $\mathcal{P}$  so that there exists a function  $\zeta: [0, \infty) \rightarrow \mathbb{N}$  such that for every point  $x \in X$  and every radius  $r$  the ball  $B_x(r)$  intersects at most  $\zeta(r)$  regions of  $\mathcal{P}$ .

Then, if  $f: X \rightarrow X$  is a  $L$ -Lipschitz map and  $Y \subset X$  is any subset, then  $f(Y)$  intersect at most  $\zeta(L \text{diam}(Y)/2)$  regions of  $\mathcal{P}$ .

## 5 Voronoi tessellations

A convenient way for defining measurable partitions on a metric space  $(X, d)$  is given by the Voronoi tessellations.

**Definition 5.1.** Given a countable discrete subset  $Y \subseteq X$  the associated *Voronoi tessellation* is the family  $\mathcal{V}(Y) := \{R(y) \mid y \in Y\}$  where

$$R(y) := \{x \in X \mid d(x, y) < d(x, y') \text{ for all } y' \in Y, y' \neq y\}.$$

If the metric is nice enough, the regions  $R(y)$  will be disjoint open sets. Assume that  $X$  is also given a measure  $\nu$  defined on the Borel subsets. In case that for any couple of points  $y \neq y'$  of  $Y$  the hyperplane  $P(y, y') = \{x \in X \mid d(x, y) = d(x, y')\}$  has measure zero, the Voronoi tessellation  $\mathcal{V}(Y)$  covers a conull subset of  $X$  and it is then a measurable partition of  $X$ .

Typical examples of well-behaved metric spaces are the Riemannian manifolds. Let  $(M, \varrho)$  be a Riemannian manifold. The volume form induced by  $\varrho$  naturally defines a measure  $\nu$  on  $M$  and every Voronoi tessellation of  $M$  forms a measurable partition of  $(M, \nu)$ . Before we proceed, we fix some notation:

**Definition 5.2.** Let  $r > 0$  be a constant. A subset  $Y$  of a metric space  $(X, d)$  is *r-separated* if  $d(y, y') \geq r$  for every couple of distinct points  $y \neq y' \in Y$ . It is *r-dense* if for every point  $x \in X$  there exists a  $y \in Y$  with  $d(x, y) < r$ .

Recall that the *injectivity radius*  $\text{inj}(M)$  of a Riemannian manifold  $(M, \varrho)$  is the largest  $r \geq 0$  so that for every point  $x \in M$  the exponential map is injective when restricted to the ball  $B_0(r) \subset T_x(r)$ . We will need the following technical lemma:

**Lemma 5.3.** *Let  $(M, \varrho)$  be a complete  $n$ -dimensional Riemannian manifold with pinched sectional curvature  $-K \leq \kappa_\varrho \leq K$  and positive injectivity radius  $\text{inj}(M) > 0$ . Fix any constant  $0 < C < 2\text{inj}(M)$ . If a discrete subset  $Y \subseteq M$  is  $r$ -separated and  $R$ -dense for some constants  $0 < r \leq R$  with  $r \leq C$ , then the Voronoi tessellation  $\mathcal{V}(Y)$  has measure ratios bounded by  $Q$ , where  $Q = Q(n, K, C, R/r)$  is a constant depending only on  $n, K, C$  and the ratio  $R/r$ .*

To prove Lemma 5.3 it is enough to realise that every Voronoi cell contains a ball of radius  $r/2$  and is contained in a ball of radius  $R$ . One can then estimate the volume of such cell using the Bishop-Gromov Comparison Theorem. We will omit the details of the argument.

*Remark 5.4.* With the same techniques one can also prove that every compact Riemannian manifold is a doubling measure space.

We remark that in any metric space for any fixed  $r > 0$  we can find a maximal  $r$ -separated subset. Such subsets are exactly those sets which are both  $r$ -separated and  $r$ -dense. Then we have:

**Theorem 5.5.** *Let  $\Gamma$  be a finitely generated group and  $S$  a finite symmetric generating set which contains the identity. Given a compact Riemannian manifold  $(M, \varrho)$  and a sequence  $r_n \rightarrow 0$ , choose for every  $n$  a maximal  $r_n$ -separated set  $Y_n$  and consider the associated Voronoi tessellation  $\mathcal{V}(Y_n)$ . If  $\Gamma$  acts by quasi-symmetric homeomorphisms on  $M$  and the action has bounded measure distortion, then the sequence of approximating graphs  $\mathcal{G}(\mathcal{V}(Y_n))$  is a family of expanders if and only if the action is expanding in measure.*

*Proof.* Since  $M$  is compact,  $\text{inj}(M) > 0$  and  $M$  has pinched sectional curvature. For all but finitely many  $n$  we have  $r_n < \text{inj}(n)$  and we can therefore apply Lemma 5.3 to deduce that the measurable partitions  $\mathcal{V}(Y_n)$  have uniformly bounded measure ratios. Moreover, the diameter of the regions  $R \in \mathcal{V}(Y_n)$  is at most  $2r_n$  because  $Y_n$  is a maximal  $r_n$  separated set.

We are now under the hypotheses of Theorem 3.10, whence we deduce that the approximating graphs  $\mathcal{G}(\mathcal{V}(Y_n))$  have a uniform lower bound on their Cheeger constant if and only if the action of  $\Gamma$  is expanding in measure.

The regions of the Voronoi tessellation  $\mathcal{V}(Y)$  have at most eccentricity 2 and we already remarked that  $(M, \nu)$  is a doubling measure space. We are then under the hypotheses of Proposition 4.1 and thus we conclude that all the graphs  $\mathcal{G}(\mathcal{V}(Y_n))$  have uniformly bounded degrees.  $\square$

## 6 Expanders, warped cones and coarse embeddings

Let  $(X, d)$  be a compact metric space and let  $\Gamma = \langle S \rangle$  act on  $X$  by homeomorphisms. Following [Roe05], we define the *warped metric*  $\delta_\Gamma$  on  $X$  as the maximal distance such that  $\delta_\Gamma(x, y) \leq d(x, y)$  and  $\delta_\Gamma(x, sx) \leq 1$  for every  $x, y \in X$  and  $s \in S$ .

*Remark 6.1.* One can show that given two points  $x, y \in X$  their warped distance  $\delta_\Gamma(x, y)$  is equal to the infimum of the sums

$$d(x_0, x_1) + 1 + d(s_1 \cdot x_1, x_2) + 1 + \cdots + d(s_{n-1} \cdot x_{n-1}, x_n)$$

where the  $x_i$ 's are points in  $X$  with  $x = x_0$  and  $y = x_n$  and the  $s_i$ 's are elements in  $S$ . Since  $X$  is locally compact, this infimum is actually a minimum.

For simplicity, we will restrict our attention to metric spaces of the form  $(M, d)$  where  $M$  is a compact Riemannian manifold and  $d$  is the distance induced by the Riemannian metric  $\varrho$ . A similar construction can be carried out for other well-behaved compact metric spaces such as finite simplicial complexes.

For  $t \geq 1$ , we consider the family of metric spaces  $(M, d^t)$  obtained rescaling the distance  $d^t := t \cdot d$ . We can then warp each of these spaces under the action of  $\Gamma$  obtaining a different family  $(M, \delta_\Gamma^t)$ . Notice that  $d^t(x, y)$  is equal to  $\delta_\Gamma^t(x, y)$  whenever one of the two (and hence both) is smaller than 1. It follows that a set  $Y \subset M$  is  $1/3$ -separated in  $(M, d^t)$  if and only if so it is in  $(M, \delta_\Gamma^t)$ .

For every  $t \geq 1$ , let  $Y_t \subset (M, \delta_\Gamma^t)$  be a maximal  $1/3$ -separated set and let  $\mathcal{X}_t$  be the simplicial graph whose set of vertices is  $Y_t$  and whose edges are the pairs of vertices with warped distance less than 2:

$$E(\mathcal{X}_t) = \{(x, y) \mid x, y \in Y_t, \delta_\Gamma^t(x, y) < 2\}.$$

The graphs  $\mathcal{X}_t$  uniformly represent the coarse structure of the metric spaces  $(M, \delta_\Gamma^t)$ . In fact, it is easy to verify that the inclusion of the vertices  $Y_t \subset M$  induces a  $(2, 1)$ -quasi-isometry between the graph  $\mathcal{X}_t$  and  $(M, \delta_\Gamma^t)$  for every  $t \geq 1$ .

Consider now the Voronoi tessellations  $\mathcal{V}(Y_t)$  of  $M$ . We would like to say that the graph  $\mathcal{G}(\mathcal{V}(Y_t))$  approximating the action  $\Gamma \curvearrowright M$  is in some sense the same graph as  $\mathcal{X}_t$ . This is not quite the case, because two vertices  $y, y' \in Y$

may be far away in the approximating graph  $\mathcal{G}(\mathcal{V}(Y_t))$  but quite close under the distance  $d^t$  and thus form an edge of  $\mathcal{X}_t$ . For convenience we will then define the *lavish approximating graphs* as the graphs  $\tilde{\mathcal{G}}(\mathcal{V}(Y_t))$  whose vertices are the elements of  $Y_t$  and such that  $(y, y')$  forms an edge if and only if

$$s \cdot \overline{R(y)} \cap \overline{R(y')} \neq \emptyset$$

for some  $s \in S$ .

The lavish approximating graph could contain many more edges than the usual approximating graph and it could also have a rather different geometry. Despite this, the lavish approximating graphs still encode well the dynamic properties of the action of  $\Gamma$ .

**Lemma 6.2.** *Let  $\Gamma$  act on a compact Riemannian manifold  $(M, \varrho)$  via quasi-symmetric homeomorphisms and with bounded measure distortion. For any sequence  $t_n > 1$  with  $t_n \rightarrow \infty$  the action of  $\Gamma$  is expanding in measure if and only if the lavish approximating graphs  $\tilde{\mathcal{G}}(\mathcal{V}(Y_{t_n}))$  form a family of expanders.*

*Proof.* Notice first that the Voronoi tessellations obtained by the set  $Y_n$  with respect to the metrics  $d$  and  $d^t$  are actually the same. Hence they yield the same approximating graphs. On  $(M, d)$ , the set  $Y_n$  is a maximal  $(1/3)t_n^{-1}$  separated set and thus we are in the hypotheses of Theorem 5.5. From this we deduce that the usual approximating graphs  $\mathcal{G}(\mathcal{V}(Y_{t_n}))$  form a family of expanders if and only if the action is expanding in measure.

Now, the graph  $\tilde{\mathcal{G}}(\mathcal{V}(Y_{t_n}))$  contains the approximating graph  $\mathcal{G}(\mathcal{V}(Y_{t_n}))$  and they both have the same set of vertices, thus the Cheeger constants of the lavish approximating graphs are at least as big as those of the ordinary approximating graphs. The graphs  $\tilde{\mathcal{G}}(\mathcal{V}(Y_{t_n}))$  have uniformly bounded degree because there is a uniform bound on the extra edges that are added to each vertex (the number of adjacent tiles can be estimated using the doubling condition and the uniform bound on the eccentricity). It follows that when the action is expanding in measure, the graphs  $\tilde{\mathcal{G}}(\mathcal{V}(Y_{t_n}))$  form a family of expanders.

For the other implication it is enough to notice that for any vertex  $y \in Y_{t_n}$  we have that the union of the adjacent regions in the lavish graph

$$A := \bigcup \{R(y') \mid (y, y') \text{ is an edge in } \tilde{\mathcal{G}}(\mathcal{V}(Y_{t_n}))\}$$

is actually contained in a neighbourhood of radius  $3t_n^{-1}$  of the union of the adjacent regions in the usual graph

$$A \subseteq N_{3t_n^{-1}} \left( \bigcup \{R(y') \mid (y, y') \text{ is an edge in } \mathcal{G}(\mathcal{V}(Y_{t_n}))\} \right).$$

Then the same proof as for Proposition 3.8 implies that if the lavish approximating graphs form a family of expanders then the action of  $\Gamma$  is expanding in measure.  $\square$

The convenience of the lavish approximating graphs comes from the following:

**Lemma 6.3.** *For every  $t \geq 1$  the lavish approximating graph  $\tilde{\mathcal{G}}(\mathcal{V}(Y_t))$  is contained in the graph  $\mathcal{X}_t$  and the inclusion is a  $(L, A)$ -quasi-isometry where the constants  $L$  and  $A$  depend only on the geometry of  $M$ .*



*Proof.* Both  $\tilde{\mathcal{G}}(\mathcal{V}(Y_t))$  and  $\mathcal{X}_t$  have  $Y_t$  as set of vertices. If  $(y, y')$  is an edge in  $\tilde{\mathcal{G}}(\mathcal{V}(Y_t))$ , by definition there must exist an element  $x \in X$  such that  $x \in \overline{R(y)}$  and  $s \cdot x \in \overline{R(y')}$ . Then

$$\delta_\Gamma^t(y, y') \leq \delta_\Gamma^t(y, x) + \delta_\Gamma^t(x, s \cdot x) + \delta_\Gamma^t(s \cdot x, y') < 2$$

thus  $(y, y')$  is also an edge of  $\mathcal{X}_t$ .

Conversely, if  $\delta_\Gamma^t(y, y') < 2$  then either we also have  $d^t(y, y') < 2$  or there exist a point  $x \in X$  with  $d^t(y, x) < 1$  and  $d^t(s \cdot x, y') < 1$ . It is hence enough to bound the distance in  $\tilde{\mathcal{G}}(\mathcal{V}(Y_t))$  of two vertices  $y, y'$  with  $d^t(y, y') < 2$ .

Picking a geodesic path  $\gamma$  in  $(M, d^t)$  between  $y$  and  $y'$  we can define a sequence of vertices  $y = y_0, y_1, \dots, y_n = y'$  by keeping track of which regions of  $\mathcal{V}(Y_t)$  are traversed by  $\gamma$ . Then each couple  $(y_i, y_{i+1})$  is an edge of  $\tilde{\mathcal{G}}(\mathcal{V}(Y_t))$ . We can bound  $n$  using the geometry of  $M$  because all the regions  $R(y_i)$  are contained in a ball of radius 3 of  $(M, d^t)$  and thus one can obtain the required uniform bound using volume estimate techniques (it is the same kind of argument needed to prove Lemma 5.3).  $\square$

Following [Roe05] we will now define the warped cones. Let  $(M, \varrho)$  be a compact Riemannian manifold, we define the open cone on  $M$  as the space  $\mathcal{O}(M) := M \times [1, \infty)$  with the metric  $d_\mathcal{O}$  induced by the Riemannian metric  $\varrho_\mathcal{O} := t^2\varrho + dt^2$  where  $dt^2$  is the standard Euclidean metric on  $\mathbb{R}$ . An action of  $\Gamma$  on  $M$  by homeomorphism induces an action by homeomorphisms on the cone  $\mathcal{O}(M)$  which fixes coordinate  $t$ .

**Definition 6.4.** The *warped cone*  $\mathcal{O}_\Gamma(M)$  of the manifold  $M$  under the action of  $\Gamma$  is the metric space  $(\mathcal{O}(M), \delta_\Gamma)$  where  $\delta_\Gamma$  is the metric obtained warping the cone metric  $d_\mathcal{O}$  of  $\mathcal{O}(M)$ .

In the previous part of this section, we have been studying the graphs  $\mathcal{X}_t$  which were quasi-isometric approximations of the metric spaces  $(M, \delta_\Gamma^t)$ . Now, one would be tempted to see them as approximations of the geometry of the level sets of the warped cone  $\mathcal{O}_\Gamma(M)$ , and indeed we have the following:

**Lemma 6.5.** *The level set  $M \times \{t\}$  with the restriction of the metric  $\delta_\Gamma$  is  $C$ -bi-Lipschitz equivalent to the metric space  $(M, \delta_\Gamma^t)$  with a constant  $C$  depending only on the geometry of  $M$ .*

*Proof.* The obvious inclusion  $(M, \delta_\Gamma^t) \hookrightarrow (M \times \{t\}, \delta_\Gamma)$  is a 1-Lipschitz map. Indeed, for every two points  $x, y \in M$  there exist a sequence  $\gamma_0, \dots, \gamma_n$  of geodesic paths  $\gamma_i: [0, 1] \rightarrow (M, d^t)$  such that  $\gamma_0(0) = x$ ,  $\gamma_n(1) = y$ ,  $\gamma_{i+1}(0) = s \cdot \gamma_i(1)$  for some  $s \in S$  and

$$\delta_\Gamma^t(x, y) = l(\gamma_0) + 1 + l(\gamma_1) + 1 + \dots + l(\gamma_n).$$

Since the metric  $d^t$  is trivially equal to the metric obtained *via* the pull-back of the Riemannian metric  $t^2\varrho + dt^2$ , the paths  $\gamma_i$  have the same length with respect to the metric  $d_\mathcal{O}$ . The action of  $\Gamma$  is the same and it still costs 1 to ‘jump’ from  $z$  to  $s \cdot z$ . Thus the path  $\gamma_0, \dots, \gamma_n$  has the same length in  $(\mathcal{O}_\Gamma(M), \delta_\Gamma)$  and hence  $\delta_\Gamma(x, y) \leq \delta_\Gamma^t(x, y)$ .

Conversely, for every  $x, y \in M \times \{t\}$  there exists a sequence  $\gamma_0, \dots, \gamma_n$  of geodesic paths  $\gamma_i: [0, 1] \rightarrow (\mathcal{O}_\Gamma(M), d)$  such that  $\gamma_0(0) = x$ ,  $\gamma_n(1) = y$ ,  $\gamma_{i+1}(0) = s \cdot \gamma_i(1)$  for some  $s \in S$  and

$$\delta_\Gamma(x, y) = l(\gamma_0) + 1 + l(\gamma_1) + 1 + \dots + l(\gamma_n).$$

Let  $\tilde{\gamma}_i$  the projection of the path  $\gamma$  into the level set  $M \times \{t\}$ . Now the path  $\tilde{\gamma}_0, \dots, \tilde{\gamma}_n$  also joins the points  $x, y$  in  $(M, \delta_\Gamma^t)$ , thus we have

$$\delta_\Gamma^t(x, y) \leq l(\tilde{\gamma}_0) + 1 + l(\tilde{\gamma}_1) + 1 + \dots + l(\tilde{\gamma}_n). \quad (2)$$

It is hence enough to bound the lengths  $l(\tilde{\gamma}_i)$ .

For  $i = 0, \dots, n$ , let  $t_i \geq 1$  denote the level of the starting point of the path  $\gamma_i$ , so that  $\gamma_i(0)$  and  $\gamma_{i-1}(1)$  lie in  $M \times \{t_i\}$ . Let  $p: (\mathcal{O}(M), d_\mathcal{O}) \rightarrow (M \times \{1\}, d_\mathcal{O}) \cong (M, d)$  denote the projection. Since the paths  $\gamma_i$  are geodesics on  $(\mathcal{O}(M), d_\mathcal{O})$ , the images of their projections  $p(\gamma_i)$  are geodesic paths in  $(M, d)$  (possibly with non-uniform speed). Denote by  $\hat{\gamma}_i: [0, \ell_i] \rightarrow (M, \varrho)$  the geodesic of  $(M, \varrho)$  obtained parametrizing  $p(\gamma_i)$  by arc length (so that  $\ell_i$  is the actual length of  $p(\gamma_i)$ ). Define the map

$$\begin{aligned} H_i: [0, \ell_i] \times [1, \infty) &\longrightarrow \mathcal{O}(M) \\ (s, t) &\longmapsto (\hat{\gamma}(s), t) \end{aligned}$$

Then the pull-back of the metric  $\varrho_\mathcal{O}$  is just a standard cone metric

$$H_i^* \varrho_\mathcal{O} = t^2 ds^2 + dt^2$$

and we can see the path  $\gamma_i$  as a geodesic path joining the points  $(0, t_i)$  and  $(\ell_i, t_{i+1})$  in this Euclidean cone.

We can now glue these cones together, obtaining a cone of total angle  $\sum_{i=1}^n \ell_i$ :

$$\bigcup_{i=1}^n ([0, \ell_i] \times [1, \infty), t^2 ds^2 + dt^2) = ([0, \sum_{i=1}^n \ell_i] \times [1, \infty), t^2 ds^2 + dt^2).$$

Let  $L := \sum_{i=1}^n \ell_i$ . Then the paths  $\gamma_i$  coincide at the gluing points, thus they can be joined to form a path between  $(0, t_0)$  and  $(L, t_0)$ . Notice that also the paths  $\tilde{\gamma}_i$  can be realised in the cone  $[0, L] \times [0, \infty)$  and they coincide with the projections of the  $\gamma_i$ 's onto the level set  $[0, L] \times \{t_0\}$ , thus we have

$$\sum_{i=1}^n l(\tilde{\gamma}_i) = Lt_0.$$

A straightforward computation in Euclidean geometry shows that the distance between  $(0, t_0)$  and  $(L, t_0)$  inside the cone  $[0, L] \times [0, \infty)$  satisfies:

$$d((0, t_0), (L, t_0)) \geq \begin{cases} 2t_0 \sin(L/2) & \text{if } L \leq \pi \\ 2t_0 & \text{if } L \geq \pi \end{cases}$$

and since  $\sin(x) \geq 2x/\pi$  for  $x \leq \pi/2$ , we deduce that

$$\sum_{i=1}^n l(\tilde{\gamma}_i) = Lt_0 \leq \max \left\{ \frac{\pi}{2}, \frac{L}{2} \right\} d((0, t_0), (L, t_0)) \leq \max \left\{ \frac{\pi}{2}, \frac{L}{2} \right\} \sum_{i=1}^n l(\ell_i).$$

To complete the proof of the lemma, it is now enough to notice that  $L$  is bounded by the diameter of  $(M, \varrho)$ , thus letting  $C := \max \{ \pi/2, \text{diam}(M, d)/2 \}$  yields  $\delta_\Gamma^t \leq C\delta_\Gamma$  on  $M \times \{t\}$ .  $\square$

We can now collect the results of this section in the following theorem. Here and after, an *unbounded sequence of level sets* of a warped cone  $\mathcal{O}_\Gamma(M)$  is a sequence of level sets  $M \times \{t_n\} \subset \mathcal{O}_\Gamma(M)$  with  $t_n \rightarrow \infty$ . Recall that by Definition 2.5 an unbounded sequence of level sets forms a family of expanders if and only if it is uniformly quasi-isometric to a family of expander graphs.

**Theorem 6.6.** *Let  $\Gamma = \langle S \rangle$  act by quasi-symmetric homeomorphisms with bounded measure distortion on a compact Riemannian manifold  $(M, \varrho)$ . Then one (equivalently, every) unbounded sequence of level sets  $M \times \{t\}$  of the warped cone  $\mathcal{O}_\Gamma(M)$  forms a family of expanders if and only if the action is expanding in measure.*

*Proof.* Fix any sequence  $t_n$  with  $t_n \rightarrow \infty$ . By Lemma 6.5, the level sets  $M \times \{t_n\}$  with the metric induced from  $\mathcal{O}_\Gamma(M)$  are all uniformly quasi-isometric to the warped metric spaces  $(M, \delta_\Gamma^{t_n})$ . Now, pick a maximal  $1/3$ -separated set  $Y_{t_n} \subset (M, \delta_\Gamma^{t_n})$  and build the graphs  $\mathcal{X}_{t_n}$  as before. As already noted, the spaces  $(M, \delta_\Gamma^{t_n})$  are uniformly quasi isometric to the graphs  $\mathcal{X}_t$  which in turns are uniformly quasi-isometric to the lavish approximating graphs  $\tilde{G}(\mathcal{V}(Y_t))$  by Lemma 6.3. Therefore, the theorem follows from Lemma 6.2.  $\square$

Theorem B can be used to provide restrictions to coarse embeddability of warped cones in Hilbert spaces.

**Corollary 6.7.** *If the action  $\Gamma \curvearrowright M$  is expanding in measure, then the warped cone  $\mathcal{O}_\Gamma(M)$  does not coarsely embed into any  $L^p$  space.*

*Remark 6.8.* Corollary 6.7 was recently proved in [NS15]. Their result is valid for a larger class of measures but it only applies to actions which are measure preserving.

## 7 A spectral criterion for measure preserving actions

Following [BFGM07] we give the following:

**Definition 7.1.** Let  $G$  be a locally compact group and  $E$  a Banach space (either real or complex). We say that an action by linear isometries  $\pi: G \curvearrowright E$  (continuous with respect to the weak operator topology) has *almost invariant vectors* if there exists a sequence  $v_n \in E \setminus \{0\}$  such that

$$\lim_{n \rightarrow \infty} \frac{\text{diam}(\pi(K)v_n)}{\|v_n\|} = 0$$

for every compact set  $K \subseteq G$ .

Let  $(X, \nu)$  be a probability space and  $\rho: \Gamma \curvearrowright X$  a measure preserving action. As always, we assume that  $\Gamma$  is finitely generated and we fix a symmetric generating  $S$  with  $1 \in S$ . For  $1 \leq p \leq \infty$  we will denote by  $L^p(X)$  is the

Banach space of complex valued functions of  $X$  with finite  $L^p$  norm (we might have used real valued functions instead: all of the following results hold in the real case as well). The action on  $X$  induces a (continuous) unitary left action  $\pi_\rho: \Gamma \curvearrowright L^p(X)$  by pre-composition:  $(g \cdot f)(x) := f(g^{-1} \cdot x)$ .

The action  $\pi_\rho: \Gamma \curvearrowright L^p(X)$  clearly has almost invariant vectors because the constant functions are genuine invariant vectors. The canonical complement of the subspace of constant functions in  $L^p$  is given by the subspace of functions with zero average

$$L_0^p(X) := \left\{ f \in L^p(X) \mid \int_X f(x) d\nu(x) = 0 \right\}.$$

The action of  $\Gamma$  preserves  $L_0^p(X)$  and we will denote again with  $\pi_\rho$  the induced action  $\pi_\rho: \Gamma \curvearrowright L_0^p(X)$ .

Notice that the action  $\pi_\rho: \Gamma \curvearrowright L_0^p(X)$  has almost invariant vectors if and only if there exists a sequence  $f_n \in L_0^p(X) \setminus \{0\}$  such that

$$\lim_{n \rightarrow \infty} \frac{\|s \cdot f_n - f_n\|_p}{\|f_n\|_p} = 0$$

for every  $s \in S$ . Equivalently,  $\pi_\rho$  does *not* have almost invariant vectors in  $L_0^p(X)$  if and only if there exists a positive constant  $\delta > 0$  so that

$$\sum_{s \in S} \|s \cdot f - f\|_p \geq \delta \|f\|_p$$

for every function  $f \in L_0^p(X)$ . In literature, an action with this property is said to have a *spectral gap* in  $L_0^p$ .

**Lemma 7.2.** *For every function  $f \in L_0^p(X)$  and every constant  $c \in \mathbb{R}$  we have*

$$\|f + c\|_p \geq \frac{\|f\|_p}{2}.$$

*Proof.* Let  $g$  be any function in  $L^p(X)$ . Applying Jensen inequality we have

$$\left( \int_X g(x) d\nu(x) \right)^p \leq \int_X g(x)^p d\nu(x).$$

Denote by  $\nu(g)$  the average  $\int_X g(x) d\nu(x)$ . Then we have:

$$\|g(x) - \nu(g)\|_p \leq \|g\|_p + \|\nu(g)\|_p \leq 2\|g\|_p. \quad (3)$$

Now, for any constant  $c$  and any  $f \in L_0^2$ , the average  $\nu(f + c)$  is equal to  $c$ . Thus inequality 3 reads as

$$\|f\|_p \leq 2\|f(x) + c\|_p.$$

□

**Proposition 7.3.** *Let  $\Gamma$  be a finitely generated group and  $S$  a finite generating set with  $S = S^{-1}$  and  $1 \in S$ . Let  $\rho: \Gamma \curvearrowright (X, \nu)$  be a measure preserving action on a probability space. Then, for any  $1 \leq p < \infty$  the induced action  $\pi_\rho: \Gamma \curvearrowright L_0^p$  does not have almost invariant vectors if and only if  $\rho$  is expanding in measure.*

*Proof.* If the action is not expanding in measure, then there exists a sequence of measurable sets  $A_n$  with measure  $\nu(A_n) \leq 1/2$  and  $\nu(S \cdot A_n)/\nu(A_n) \rightarrow 1$ . Looking at the symmetric difference, we have that  $\nu((s \cdot A_n) \triangle A_n)/\nu(A_n) \rightarrow 0$  for every  $s \in S$ . Denote by  $\mathbb{1}_{A_n}$  the indicator function of the set  $A_n$  and let  $f_n(x) := \mathbb{1}_{A_n}(x) - \nu(A_n)$ . The sequence  $\{f_n\}_{n \in \mathbb{N}}$  lies in  $L_0^p(X)$  and we have

$$\|s \cdot f_n - f_n\|_p^p = \nu(A_n \setminus s \cdot A_n) + \nu(s \cdot A_n \setminus A_n) = \nu((s \cdot A_n) \triangle A_n)$$

while

$$\begin{aligned} \|f_n\|_p^p &= \nu(A_n)(1 - \nu(A_n))^p + (1 - \nu(A_n))\nu(A_n)^p \\ &\geq \nu(A_n)(1 - \nu(A_n))^p \\ &\geq \frac{1}{2^p} \nu(A_n). \end{aligned}$$

It follows that  $(f_n)$  is a sequence of almost invariant vectors in  $L_0^p$ .

For the converse implication, fix any  $1 \leq p < \infty$ . We need to show that if  $\rho$  is  $\varepsilon$ -expanding then there is a constant  $\delta > 0$  so that for every function  $f \in L_0^p(X)$  we have  $\sum_{s \in S} \|s \cdot f - f\|_p \geq \delta \|f\|_p$ . We can restrict our attention to real functions because  $\Gamma$  acts separately on the real and imaginary parts. By density, it is then enough to prove the statement for scale functions of the form

$$f(x) = \sum_{i=0}^N \alpha_i \mathbb{1}_{A_i}(x)$$

with  $\alpha_i \in \mathbb{R}$  and  $A_N \subseteq A_{N-1} \subseteq \dots \subseteq A_0$ .

There exists a constant  $c$  such that both the set  $\{x \mid f(x) > c\}$  and  $\{x \mid f(x) < c\}$  have measure smaller or equal than  $1/2$ . Let  $g := f - c$ , then by Lemma 7.2 we have that  $\|g\|_p \geq \|f\|_p/2$ . Changing sign if necessary, we may assume that  $\|g^+\|_p \geq \frac{1}{4}\|f\|_p$  where  $g^+ = \max\{g, 0\}$ .

Clearly  $\|s \cdot f - f\|_p = \|s \cdot g - g\|_p \geq \|s \cdot g^+ - g^+\|_p$ , thus we only need to find a lower bound for the latter. The function  $g^+$  is still a scale function

$$g^+(x) = \sum_{i=0}^n \beta_i \mathbb{1}_{B_i}(x)$$

with  $B_{i+1} \subseteq B_i$ , but this time we can also assume  $\beta_i > 0$  and  $\nu(B_i) \leq 1/2$  for every  $i = 0, \dots, n$ .

Since the  $B_i$ 's are nested, we have

$$\begin{aligned} |s \cdot g^+ - g^+|(x) &\geq \sum_{i=0}^n \beta_i |\mathbb{1}_{B_i}(s^{-1}x) - \mathbb{1}_{B_i}(x)| \\ &\geq \sum_{i=0}^n \beta_i (\mathbb{1}_{B_i \cup sB_i}(x) - \mathbb{1}_{B_i}(x)) \\ &= h_s(x) - g^+(x) \end{aligned}$$

where

$$h_s(x) := \sum_{i=0}^n \beta_i \mathbb{1}_{B_i \cup sB_i}(x).$$

Note that

$$\sum_{s \in S} \|h_s\|_p^p = \int_{\mathbb{R}^+} \sum_{s \in S} \nu(\{x \mid h_s(x)^p \geq r\}) dr.$$

Since the action is expanding we have

$$\sum_{s \in S} \nu(\{x \mid h_s(x)^p \geq r\}) \geq (|S| + \varepsilon) \nu(\{x \mid g^+(x)^p \geq r\});$$

thus we get:

$$\sum_{s \in S} \|h_s\|_p^p \geq \int_{\mathbb{R}^+} (|S| + \varepsilon) \nu(\{x \mid g^+(x)^p \geq r\}) dr = (|S| + \varepsilon) \|g^+\|_p^p$$

whence we deduce that there exists  $s \in S$  such that  $\|h_s\|_p \geq (1 + \varepsilon/|S|)^{1/p} \|g^+\|_p$ .

Let  $\delta' := (1 + \varepsilon/|S|)^{1/p} - 1$ , then for the same  $s \in S$  we have

$$\|s \cdot g^+ - g^+\|_p \geq \|h_s - g^+\|_p \geq \|h_s\|_p - \|g^+\|_p \geq \delta' \|g^+\|_p.$$

and the Lemma follows because

$$\|s \cdot f - f\|_p \geq \|s \cdot g^+ - g^+\|_p \geq \delta' \|g^+\|_p \geq \frac{\delta'}{4} \|f\|_p.$$

□

Combining Proposition 7.3 with Theorem 5.5 we obtain the following:

**Corollary 7.4.** *Let  $\Gamma = \langle S \rangle$  act by quasi-symmetric homeomorphisms on a compact Riemannian manifold  $(M, \varrho)$  and assume that the action preserves the Riemannian measure. Then one (any) unbounded sequence of level sets of  $\mathcal{O}_\Gamma(M)$  forms a family of expanders if and only if the action has a spectral gap.*

*Remark 7.5.* Let  $\Gamma_i$  be a decreasing sequence of finite index subgroups of  $\Gamma = \langle S \rangle$ . Endow the set of left cosets  $\Gamma/\Gamma_i$  with the uniform probability measure. Then  $\Gamma$  acts on  $\Gamma/\Gamma_i$  by multiplication and by considering the complete partition we clearly have that the approximating graphs are actually equal to the Schreier graphs of  $\Gamma/\Gamma_i$  with respect to the set  $S$ . It follows that Proposition 7.3 and Lemma 3.5 recover the well-known fact that those Schreier graphs form a family of expanders if and only if  $\Gamma$  has property  $(\tau)$  with respect to the sequence of  $\Gamma_i$ .

*Remark 7.6.* It follows from Proposition 7.3 that the existence of almost invariant vectors for the action  $\pi_p$  does not depend on what  $1 \leq p < \infty$  is being considered. This was proved in the more general setting of measure-class preserving actions of topological groups in [BFGM07, Remark 4.3]. They state it for real  $L^p$  spaces, but their proof works in the complex case as well.

*Remark 7.7.* Proposition 7.3 also implies that a measure preserving action  $\Gamma \curvearrowright X$  is expanding if and only if there exists a unique invariant mean on  $L^\infty(X)$  (see [Ros81] and [Sch81]). Therefore, the expansion in measure is equivalent to a positive answer to the Ruziewicz problem for this action.

## 8 A source of expanding actions: subgroups generated by Kazhdan sets

In this section we will use the spectral criterion from Section 7 to link expanding actions to the well developed machinery of Kazhdan's property (T). Such link proves to be an invaluable tool in constructing explicit examples of expanding actions and it also allows us to reformulate some classical theorems and conjectures.

We begin by recalling some definitions. Here  $G$  will always be a locally compact second countable Hausdorff topological group and we will only consider its continuous unitary representations. Most of the tools we use can be found in [Sha00] and [BdlHV08].

**Definition 8.1.** Let  $G$  be a locally compact topological group. Let  $K \subseteq G$  be a compact subset and  $\varepsilon > 0$  a constant. If  $\mathcal{H}$  is a (complex) Hilbert space and  $\pi: G \rightarrow U(\mathcal{H})$  is a unitary representation, a  $(K, \varepsilon)$ -invariant vector is a vector  $v \in \mathcal{H}$  such that  $\|\pi(g)v - v\| \leq \varepsilon\|v\|$  for every  $g \in K$ .

Given a family  $\mathcal{F}$  of unitary representations of  $G$ , we say that a pair  $(K, \varepsilon)$  is a *Kazhdan pair* for  $\mathcal{F}$  if every representation  $\pi \in \mathcal{F}$  does not have non-zero  $(K, \varepsilon)$ -invariant vectors. When this is the case,  $K$  (resp.  $\varepsilon$ ) is a *Kazhdan set* (resp. a *Kazhdan constant*) for  $\mathcal{F}$ .

It is easy to verify that a representation  $\pi: G \rightarrow U(\mathcal{H})$  admits a Kazhdan pair if and only if it does not admit a sequence of almost invariant vectors. One can also show that if a family  $\mathcal{F}$  of unitary representations of  $G$  admits a Kazhdan pair  $(K, \varepsilon)$  and  $K'$  is any compact generating set for the group  $G$ , then  $K'$  is a Kazhdan set for  $\mathcal{F}$  (for an appropriate Kazhdan constant  $\varepsilon'$ . See [BdlHV08, Chapter 1]).

**Definition 8.2.** A group  $G$  is said to have *property (T)* if the family  $\mathcal{U}_0$  of all continuous unitary representations without non-trivial invariant vectors admits a Kazhdan pair  $(K, \varepsilon)$ . Such pair  $(K, \varepsilon)$  (resp. set, constant) is a *Kazhdan pair* (resp. *set*, *constant*) of the group  $G$ . In the remainder, when we say that a set  $K$  is a Kazhdan set without specifying any family of representations we mean that  $K$  is a Kazhdan set of the group.

*Remark 8.3.* If a compact subset  $K \subseteq G$  is a Kazhdan set for every unitary  $G$ -representation  $\pi$  with no invariant vector, then it is a Kazhdan set of  $G$ . Indeed, if  $\varepsilon_\pi > 0$  is the largest constant such that  $(K, \varepsilon_\pi)$  is a Kazhdan pair for  $\pi$ , then there must exist a  $\varepsilon > 0$  such that  $\varepsilon_\pi \geq \varepsilon$  for every  $\pi$ ; otherwise one would get a contradiction by considering the direct sum of a sequence of representations  $\pi_n \in \mathcal{U}_0$  with  $\varepsilon_{\pi_n} \rightarrow 0$ .

Recall that the *diagonal matrix coefficients* of a unitary representation  $\pi: G \rightarrow U(\mathcal{H})$  are the complex functions on  $G$  sending  $g \mapsto \langle \pi(g)v, v \rangle$  where  $v$  is any fixed vector in  $\mathcal{H}$ . Given two unitary representations  $\pi, \pi'$  of  $G$ ,  $\pi'$  is *weakly contained in*  $\pi$  (denoted  $\pi' \prec \pi$ ) if every diagonal matrix coefficient of  $\pi'$  can be approximated uniformly on compact sets by convex combinations of matrix coefficients of  $\pi$ .

Let  $I_G$  denote the trivial representation of  $G$ . Then it is easy to check a unitary representation  $\pi$  admits a Kazhdan pair if and only if  $\pi$  does *not* weakly contain the trivial representation ( $I_G \not\prec \pi$ ). In particular, a group  $G$  has

property (T) if and only if every unitary representation  $\pi$  weakly containing  $I_G$  has non-trivial invariant vectors (*i.e.* it contains  $I_G$  as a subrepresentation).

Finally, given a probability measure  $\mu$  on  $G$  and a unitary representation  $\pi: G \rightarrow \mathcal{U}(\mathcal{H})$ , the  $\mu$ -convolution operator  $\pi(\mu): \mathcal{H} \rightarrow \mathcal{H}$  is defined as the operator such that

$$\langle \pi(\mu)v, u \rangle = \int_G \langle \pi(g)v, u \rangle d\mu$$

for every  $v, u \in \mathcal{H}$ .

Note that since  $\mu$  is a probability measure  $\|\pi(\mu)\| \leq 1$ . Moreover, if  $S \subset G$  is a finite symmetric set with  $1 \in S$ , and  $\mu_S := \frac{1}{|S|} \sum_{s \in S} \delta_s$  is the probability measure equidistributed on  $S$ , then one can show using elementary spectral theory that the representation  $\pi$  has a spectral gap (in the sense of Section 7) if and only if  $\|\pi(\mu_S)\| < 1$  and this is equivalent to  $r_{sp}\pi(\mu_S) < 1$ , where  $r_{sp}\pi(\mu_S)$  is the *spectral radius* of the operator  $\pi(\mu_S)$ . In particular,  $\|\pi(\mu_S)\| < 1$  if and only if  $S$  is a Kazhdan set for  $\pi$ .

We can now state the motivating result of this section and explain some of its applications. Given a finite subset  $S \subseteq G$  that is symmetric and contains the identity, denote by  $\Gamma := \langle S \rangle$  the subgroup of  $G$  generated by  $S$ . If  $(X, \nu)$  is a probability space and  $\rho: G \curvearrowright X$  is a measure preserving action, we can investigate expansion properties of the restriction of  $\rho$  to  $\Gamma$  and we obtain the following:

**Proposition 8.4.** *The restriction  $\rho|_\Gamma: \Gamma \curvearrowright (X, \nu)$  is expanding in measure if and only if there exists a constant  $\varepsilon > 0$  such that  $(S, \varepsilon)$  is a Kazhdan pair for the representation  $\pi_\rho: G \curvearrowright L_0^2(X)$ .*

*Proof.* By Proposition 7.3 we know that  $\rho|_\Gamma$  is expanding in measure if and only if  $\pi_{\rho|_\Gamma}: \Gamma \curvearrowright L_0^2(X)$  does not have almost invariant vectors. That is, if and only if  $\pi_{\rho|_\Gamma}$  admits a Kazhdan pair  $(K, \varepsilon)$ . Since  $S$  is a finite generating set of  $\Gamma$ , such a Kazhdan pair exists if and only if there exists a constant  $\varepsilon' > 0$  so that  $(S, \varepsilon')$  is itself a Kazhdan pair for  $\pi_{\rho|_\Gamma}$ .

Now, since  $\pi_{\rho|_\Gamma} = (\pi_\rho)|_\Gamma$  and  $S \subseteq \Gamma$ , we have that  $(S, \varepsilon')$  is a Kazhdan pair for  $\pi_{\rho|_\Gamma}$  if and only if it is a Kazhdan pair for  $\pi_\rho$  as well.  $\square$

*Remark 8.5.* When  $(S, \varepsilon)$  is a Kazhdan pair for  $\pi_\rho$ , one can retrieve explicit bounds on the expansion constant of  $\rho$  in term of the Kazhdan constant  $\varepsilon$  and vice versa following the proof of Proposition 7.3.

In the rest of this section we describe some consequences of Proposition 8.4 (we refer the reader to [CG11, Section 2] for more examples of actions of finitely generated groups on measure spaces that have a spectral gap).

**A characterisation of Kazhdan sets.** Schmidt, Connes and Weiss characterised groups with Kazhdan's property (T) in term of their ergodic actions. Specifically, they proved that  $G$  has property (T) if and only if for every measure preserving ergodic action on a probability space  $\rho: G \curvearrowright (X, \nu)$  the induced unitary representation  $\pi_\rho: G \curvearrowright L_0^2(X)$  admits a Kazhdan pair.

Using Remark 8.3, one can adapt the proof of the Schmidt-Connes-Weiss theorem given in [BdlHV08, Theorem 6.3.4] to prove the following more precise statement: a compact subset  $K \subseteq G$  is a Kazhdan set of  $G$  if and only if for



every ergodic action  $\rho$  of  $G$  the set  $K$  is a Kazhdan set for the representation  $\pi_\rho$ . Therefore, Proposition 8.4 implies the following:

**Theorem 8.6.** *Let  $S \subset G$  be a finite symmetric set containing the identity and let  $\Gamma := \langle S \rangle \subset G$ . Then,  $S$  is a Kazhdan set of  $G$  if and only if the restriction to  $\Gamma$  of every ergodic action  $\rho: G \curvearrowright (X, \nu)$  is expanding in measure.*

*Moreover, when  $S$  is a Kazhdan set of  $G$ , all the  $\Gamma$ -actions obtained as restrictions of ergodic  $G$ -actions share a lower bound on their expansion constants depending only on the Kazhdan constant of  $S$  in  $G$ .*

**Non-compact Lie groups.** In the setting of non-compact Lie groups, the work of Y. Shalom provides very general means of proving spectral gap properties. In particular, he proved the following.

**Theorem 8.7** ([Sha00], Theorem C). *Let  $G = \prod G_i$  be a semisimple Lie group with finite centre and let  $\pi$  be a unitary  $G$ -representation and  $\mu$  a probability measure on  $G$ . If either of the following is true:*

- a)  $I_G \not\prec \pi|_{G_i}$  for every  $G_i$  and  $\mu$  is not supported on a closed amenable subgroup of  $G$ ,
- b)  $I_G \not\prec \pi$  and for every  $i$  the projection of  $\mu$  on  $G_i$  is not supported on a closed amenable subgroup of  $G_i$ ,

*then the operator  $\mu(\pi)$  has spectral radius strictly less than one  $r_{sp}(\mu) < 1$ .*

As a corollary one can produce a multitude of examples of expanding actions. Indeed, let  $\rho: G \curvearrowright (X, \nu)$  be a measure preserving action of a semisimple Lie group with finite centre and let  $\pi_\rho: G \curvearrowright L_0^2(X)$  be the induced unitary representation. If  $I_G \not\prec \pi|_{G_i}$  for every  $G_i$  (resp.  $I_G \not\prec \pi$ ) and  $S$  is any finite symmetric set containing the identity and so that the closure of  $\Gamma := \langle S \rangle$  in  $G$  is not amenable (resp. the closures of the projections of  $\Gamma$  to the  $G_i$ 's are not amenable), then the restriction  $\rho|_\Gamma: \Gamma \curvearrowright (X, \mu)$  is expanding in measure by Proposition 8.4.

For example, if a simple Lie group  $G$  has Kazhdan property (T) and finite centre and  $\rho: G \curvearrowright (X, \nu)$  is any measure preserving ergodic action then  $I_G \not\prec \pi|_G$ . If  $S$  generates a discrete subgroup  $\Gamma < G$ , then  $\rho|_\Gamma$  is expanding as soon as  $\Gamma$  is not amenable (*i.e.* as soon as  $\Gamma$  contains a non-abelian free subgroup). A typical example of ergodic action of  $G$  is the action by left multiplication  $G \curvearrowright G/\Lambda$  where  $\Lambda < G$  is a lattice. More generally, if  $G < G'$  where  $G'$  is a finite product of connected, non compact, simple Lie groups with finite centre and  $\Lambda$  is any irreducible lattice of  $G'$ , then Moore's Ergodicity Theorem implies that the action by left multiplication  $G \curvearrowright G'/\Lambda$  is ergodic if and only if the closure of  $G$  in  $G'$  is not compact.

*Remark 8.8.* Any right invariant Riemannian metric on  $G'$  descends to a Riemannian metric on  $G'/\Lambda$  whose volume form is (a multiple of) the restriction of the Haar measure and the action on the left  $G' \curvearrowright G'/\Lambda$  is by bilipschitz diffeomorphisms. In the example above, it follows that if the irreducible lattice  $\Lambda$  is also uniform, then one can consider the warped cone  $\mathcal{O}(G'/\Lambda)$  and by Corollary 7.4 any unbounded sequence of its level sets forms a family of expanders.

*Remark 8.9.* In the above example, we assumed  $G$  to have property (T), but the subgroup  $\Gamma$  does not need to have it (nor does  $G'$ ). Indeed, taking  $\Gamma$  to be any discrete non-abelian free group would do. This is a very interesting feature, as it allows us to build expanders out of actions of free groups and, more generally, of a-T-menable groups. Moreover, we can produce examples of warped cones that do not coarsely embed into Hilbert spaces even if the warping group  $\Gamma$  has Haagerup property.

In the same paper, Shalom constructed explicitly finite Kazhdan sets for algebraic groups and he was also able to compute their Kazhdan constants [Sha00, Theorem A]. More precisely, he finds Kazhdan sets of  $m$  elements whose Kazhdan constant is

$$\varepsilon = \sqrt{2 - 2(\sqrt{2m-1}/m)}$$

(and we already remarked that these estimates immediately translate in estimates for the Cheeger constants of the approximating graphs). As a concrete example, he proves that the matrices

$$\left( \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & I_{n-2} \end{array} \right), \left( \begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \hline 0 & 0 & I_{n-2} \end{array} \right)$$

form a Kazhdan set of two elements for  $\mathrm{Sl}_n(\mathbb{R})$  for every  $n \geq 3$ .

**Compact Lie groups.** Theorem 8.7 can only be applied to non-compact Lie groups. Still, the case of compact Lie groups is all but devoid of interest. In fact, one can show that every simple, connected, compact Lie group admits finite Kazhdan sets (see [Sha99, Section 5] for more on this).

Explicit examples are provided by Bourgain and Gamburd in [BG07]. There they prove that if  $k$  elements  $g_1, \dots, g_k \in \mathrm{SU}(2)$  generate a free subgroup of  $\mathrm{SU}(2)$  and they satisfy a non-abelian diophantine property then they form a Kazhdan set of  $\mathrm{SU}(2)$ . In particular, they show that when two matrices with algebraic entries  $a, b \in \mathrm{SU}(2) \cap \mathrm{GL}_2(\overline{\mathbb{Q}})$  freely generate a free group  $\Gamma < \mathrm{SU}(2)$ , then every ergodic action of  $\mathrm{SU}(2)$  restricts to an expanding action of  $\Gamma$  (here the generating set we should use is actually  $S = \{1, a^\pm, b^\pm\}$ ).

An obvious example of an ergodic action of  $\mathrm{SU}(2)$  on a compact space is the action by left multiplication of  $\mathrm{SU}(2)$  on itself. Alternatively, note that  $\mathrm{SU}(2)$  is the double cover of  $\mathrm{SO}(3)$  and the action of the latter on the sphere  $S^2$  is ergodic. Thus, we obtain the following:

**Corollary 8.10.** *Let  $a$  and  $b$  be two independent rotations of  $S^2$  whose matrices have algebraic entries and let  $F_2 = \langle a, b \rangle$  be the generated subgroup of  $\mathrm{SO}(3)$ . Then any unbounded sequence of level sets of the warped cone  $\mathcal{O}_{F_2}(S^2)$  forms a family of expanders.*

*Remark 8.11.* The existence of actions by rotations on the sphere  $S^2$  that are expanding in measure has already been successfully used in relation to the Ruziewicz problem and to the problem of constructing finite equidistributed subsets of  $S^2$  [Lub10].

**A conjecture of Gamburd, Jakobson and Sarnak.** The results of [BG07] have been later extended to  $\mathrm{SU}(n)$  for any  $n \geq 2$  in [BG10] and subsequently to all compact simple Lie groups in [BdS14]. These works build on

the notion of *non-abelian diophantine property* introduced in [GJS99] in order to study spectral gap properties for generic subgroups of rotations.

For a generic  $k$ -tuple of elements in  $SU(2)$ , it is known that the action on  $S^2$  of the generated subgroup  $\Gamma < SU(2)$  is ergodic; and it is conjectured in [GJS99] that the action of  $\Gamma$  should also have a spectral gap (which is a much stronger property). A partial result is due to Fisher [Fis06] who managed to prove that if the conjecture is false then the set of  $k$ -tuples inducing actions with spectral gap must have null measure. It is unknown whether the group generated by a generic  $k$ -tuple has the non-abelian diophantine property (an affirmative answer to the latter would clearly imply the conjecture).

More generally, it is unknown whether the action by left multiplication  $\Gamma \curvearrowright G$  of a generic finitely generated dense subgroup  $\Gamma$  of a compact simple Lie group  $G$  has a spectral gap. Corollary 7.4 immediately implies that the last conjecture is equivalent to the statement that generic warped cones form families of expanders:

**Theorem 8.12.** *A generic dense subgroup  $\Gamma$  of a compact simple Lie group  $G$  has a spectral gap if and only if one (any) unbounded sequences of the level sets of a generic warped cone  $\mathcal{O}_\Gamma(G)$  forms a family of expanders.*

**Warped cones and finite Kazhdan sets of compact groups.** As a last note, we wish to report a nice feature of compact Lie group already noted in [Sha99]. Let  $G$  be a compact Lie group with Haar measure  $\nu$  and consider its action on itself by left multiplication. The induced unitary representation  $\pi_R: G \rightarrow U(L^2(G))$  is called the *(left) regular representation* of  $G$ . Notice that the regular representation decompose as the direct sum of its restriction to  $L_0^2(G)$  and the trivial representation. The Peter-Weyl theorem implies the following strong version of Theorem 8.6:

**Lemma 8.13.** *A finite set  $S \subset G$  is a Kazhdan set of  $G$  if and only if the restriction of  $\pi_R|_{L_0^2(G)}$  to the generated group  $\Gamma = \langle S \rangle$  has a spectral gap.*

**Corollary 8.14.** *Let  $G$  be a compact Lie group and let  $\Gamma = \langle S \rangle$  where  $S \in G$  is finite symmetric subset. Choose any Riemannian metric on  $G$ . Then the level sets of the warped cone  $\mathcal{O}_\Gamma(G)$  form a family of expanders if and only if  $S$  is a Kazhdan set of  $G$ .*

*Proof.* The statement is clearly true if the volume form induced by the Riemannian metric coincides with the Haar measure  $\nu$ . Any other Riemannian volume form equals  $f(x)\nu(x)$  for some strictly positive smooth function  $f$ . Since  $G$  is a compact, there are constants  $0 < c < C$  so that  $c < f(x) < C \forall x \in G$  and it follows that  $\Gamma \curvearrowright G$  is expanding in measure with respect to  $\nu$  if and only if it is expanding in measure with respect to  $f(x)\nu(x)$ .  $\square$

## References

- [BdlHV08] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's Property (T)*, New Mathematical Monographs, Cambridge University Press, 2008.
- [BdS14] Yves Benoist and Nicolas de Saxcé, *A spectral gap theorem in simple Lie groups*, arXiv preprint arXiv:1405.1808 (2014).

- [Bek03] M. Bachir Bekka, *Kazhdan's Property (T) for the unitary group of a separable Hilbert space*, Geometric & Functional Analysis GAFA **13** (2003), no. 3, 509–520.
- [BFGM07] Uri Bader, Alex Furman, Tsachik Gelander, and Nicolas Monod, *Property (T) and rigidity for actions on Banach spaces*, Acta mathematica **198** (2007), no. 1, 57–105.
- [BG07] Jean Bourgain and Alex Gamburd, *On the spectral gap for finitely-generated subgroups of  $SU(2)$* , Inventiones mathematicae, vol. 171, issue 1, pp. 83–121 **171** (2007), 83–121.
- [BG10] Jean Bourgain and Alexander Gamburd, *Spectral gaps in  $SU(d)$* , Comptes Rendus Mathematique **348** (2010), no. 11, 609–611.
- [BY13] Jean Bourgain and Amir Yehudayoff, *Expansion in  $SL_2(\mathbb{R})$  and monotone expanders*, Geometric and Functional Analysis **23** (2013), no. 1, 1–41.
- [CG11] Jean-Pierre Conze and Yves Guivarc'h, *Ergodicity of group actions and spectral gap, applications to random walks and markov shifts*, arXiv preprint arXiv:1106.3248 (2011).
- [dC08] Yves de Cornulier, *Dense subgroups with Property (T) in Lie groups*, Commentarii Mathematici Helvetici **83** (2008), no. 1, 55–65.
- [DGLY02] A. N. Dranishnikov, G. Gong, V. Lafforgue, and G. Yu, *Uniform embeddings into Hilbert space and a question of Gromov*, Canadian Mathematical Bulletin **45** (2002), no. 1, 60–70.
- [DN15] Cornelia Druţu and Piotr W. Nowak, *Kazhdan projections, random walks and ergodic theorems*, arXiv preprint arXiv:1501.03473 (2015).
- [Enf70] Per Enflo, *On a problem of Smirnov*, Arkiv för Matematik **8** (1970), no. 2, 107–109.
- [Fis06] David Fisher, *Out( $F_n$ ) and the spectral gap conjecture*, International Mathematics Research Notices **2006** (2006), 26028.
- [GG81] Ofer Gabber and Zvi Galil, *Explicit constructions of linear-sized superconcentrators*, Journal of Computer and System Sciences **22** (1981), no. 3, 407–420.
- [GJS99] Alex Gamburd, Dmitry Jakobson, and Peter Sarnak, *Spectra of elements in the group ring of  $SU(2)$* , Journal of the European Mathematical Society **1** (1999), no. 1, 51–85.
- [GMP16] Łukasz Grabowski, András Máthé, and Oleg Pikhurko, *Measurable equidecompositions for group actions with an expansion property*, arXiv preprint arXiv:1601.02958 (2016).
- [Gro93] Mikhael Gromov, *Geometric group theory*, London Mathematical Society Lecture Note Series, vol. 2, ch. Asymptotic invariants of infinite groups, pp. 1–295, Cambridge University Press, 1993.

- [Gro03] Mikhail Gromov, *Random walk in random groups*, Geometric and Functional Analysis **13** (2003), no. 1, 73–146.
- [Hig99] Nigel Higson, *Counterexamples to the coarse Baum-Connes conjecture*.
- [HLS02] Nigel Higson, Vincent Lafforgue, and Georges Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geometric and Functional Analysis **12** (2002), no. 2, 330–354.
- [HLW06] Shlomo Hoory, Nathan Linial, and Avi Wigderson, *Expander graphs and their applications*, Bulletin of the American Mathematical Society **43** (2006), no. 4, 439–561.
- [Lub10] Alex Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Springer Science & Business Media, 2010.
- [Lub12] Alexander Lubotzky, *Expander graphs in pure and applied mathematics*, Bulletin of the American Mathematical Society **49** (2012), no. 1, 113–162.
- [Mar73] G.A. Margulis, *Explicit constructions of expanders*, Problemy Peredaci Informacii **9** (1973), no. 4, 71–80.
- [Mat97] Jiří Matoušek, *On embedding expanders into  $\ell_p$  spaces*, Israel Journal of Mathematics **102** (1997), no. 1, 189–197.
- [NS15] Piotr W. Nowak and Damian Sawicki, *Warped cones and spectral gaps*, arXiv preprint arXiv:1509.04921 (2015).
- [Oza04] Narutaka Ozawa, *A note on non-amenability of  $B(\ell_p)$  for  $p=1, 2$* , International Journal of Mathematics **15** (2004), no. 06, 557–565.
- [Pin73] Mark S. Pinsky, *On the complexity of a concentrator*, 7th International Teletraffic Conference, vol. 4, 1973, pp. 318/1–318/4.
- [Roe05] John Roe, *Warped cones and property A*, Geometry & Topology **9** (2005), no. 1, 163–178.
- [Ros81] Joseph Rosenblatt, *Uniqueness of invariant means for measure-preserving transformations*, Transactions of the American Mathematical Society **265** (1981), no. 2, 623–636.
- [Sch81] Klaus Schmidt, *Amenability, Kazhdan’s property (T), strong ergodicity and invariant means for ergodic group-actions*, Ergodic Theory and Dynamical Systems **1** (1981), no. 02, 223–236.
- [Sha97] Yehuda Shalom, *Expanding graphs and invariant means*, Combinatorica **17** (1997), no. 4, 555–575.
- [Sha99] ———, *Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan’s property (T)*, Transactions of the American Mathematical Society **351** (1999), no. 8, 3387–3412.

- [Sha00] ———, *Explicit kazhdan constants for representations of semisimple and arithmetic groups*, Annales de l’institut Fourier, vol. 50, 2000, pp. 833–863.
- [Yu00] Guoliang Yu, *The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Inventiones Mathematicae **139** (2000), no. 1, 201–240.
- [Zim13] Robert J. Zimmer, *Ergodic theory and semisimple groups*, vol. 81, Springer Science & Business Media, 2013.